

**A Brief discussion of the Seesaw Mechanism
based on a paper by
Carlo Giunti and Marco Lavender:
[hep-ph/03100238]**

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For the Neutrino Journal Club

The Dirac Equation in Hamiltonian and Lagrangian form for an electron of mass m is written as:

$$H = \gamma_4 m + \gamma_\mu \vec{\gamma} \cdot \vec{p}$$

$$\mathcal{L} = i \bar{\psi} \gamma_\mu \partial_\mu \psi - m \bar{\psi} \psi$$

\vec{p} is an operator : $i\hbar \frac{\partial}{\partial_x}$

To satisfy Lorenz invariance, the wave functions and operators are four dimensional.
That is, in general:

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\psi' = (\psi_1^* \psi_2^* \psi_3^* \psi_4^*)$$

The adjoint spinor is :

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

$$\bar{\psi} = \psi' \gamma^0 = (\psi_1^* \psi_2^* \psi_3^* \psi_4^*) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = (\psi_1^* + \psi_2^* - \psi_3^* - \psi_4^*)$$

Then :

$$\bar{\psi} \psi = (\psi_1^* + \psi_2^* - \psi_3^* - \psi_4^*) \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = (|\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 +$$

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\partial_\mu\psi - m\bar{\psi}\psi$$

The first step is to note mass term in the Lagrangian. For a massless particle, the Lagrangian can be written as:

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\partial_\mu\psi$$

Because the neutrino has mass, we must now include the mass term in the Lagrangian. That describes the electro-weak interaction. We must also account for the fact that nature Provides us with a left handed Dirac neutrino and a right handed Dirac anti-neutrino. This Really needs a discussion of parity conservation and the V-A theory, but we'll just assume All of that for this discussion. The Lagrangian mass terms for the Dirac neutrino then take the form:

$$\nu = \nu_L + \nu_R \quad \bar{\nu} = \bar{\nu}_L + \bar{\nu}_R$$

$$\mathcal{L}^D = -m_D \bar{\nu} \nu = -m_D (\bar{\nu}_R \nu_L + \bar{\nu}_L \nu_R) ; \bar{\nu}_L \nu_L = 0 ; \bar{\nu}_R \nu_R = 0$$

The charge conjugation operator is defined as :

$$C\gamma^{\mu T}C^{-1} = -\gamma^\mu$$

C operating on a particle wave function will change it to the anti particle wave function by changing reversing the charge and spin of the particle.

$$C = i\gamma^2\gamma^0 = -C^{-1} = -C' = -C^T$$

$$\nu_L^c = C(\bar{\nu}_L)^T = C\gamma^{0T}(\nu_L)^*$$

The Majorana condition states that the neutrino is the same particle as its anti - neutrino

$$\nu_R = \nu_L^c$$

$$\nu_L = \nu_R^c$$

This introduces new mass terms into the EW Lagrangian :

$$\mathcal{L}_L^M = -\frac{1}{2}m_R(\bar{\nu}_L^c\nu_L + \bar{\nu}_L\nu_L^c) ; \nu_R = \nu_L^c$$

$$\mathcal{L}_R^M = -\frac{1}{2}m_R(\bar{\nu}_R^c\nu_R + \bar{\nu}_R\nu_R^c) ; \nu_L = \nu_R^c$$

When we combine the Dirac and Majorana terms, we get a sum of four terms :

$$\begin{aligned} \mathcal{L}^{D+M} &= \mathcal{L}^D + \mathcal{L}_L^M + \mathcal{L}_R^M = \\ &-\frac{1}{2}m_D(\bar{\nu}_R\nu_L + \bar{\nu}_L\nu_R) - \frac{1}{2}m_D(\bar{\nu}_R\nu_L + \bar{\nu}_L\nu_R) - \frac{1}{2}m_R(\bar{\nu}_R^c\nu_R + \bar{\nu}_R\nu_R^c) - \frac{1}{2}m_R(\bar{\nu}_L^c\nu_L + \bar{\nu}_L\nu_L^c) \end{aligned}$$

$$\mathcal{L}^D = -m_D \bar{\nu} \nu = -m_D (\bar{\nu}_R \nu_L + \bar{\nu}_L \nu_R) ; \bar{\nu}_L \nu_L = 0 ; \bar{\nu}_R \nu_R = 0$$

The Majorana condition states that the neutrino is the same particle as its anti - neutrino.

$$\mathcal{L}_L^M = -\frac{1}{2} m_R (\bar{\nu}_L^c \nu_L + \bar{\nu}_L \nu_L^c) ; \nu_R = \nu_L^c \text{ This is the Majorana condition.}$$

$$\mathcal{L}_R^M = -\frac{1}{2} m_R (\bar{\nu}_R^c \nu_R + \bar{\nu}_R \nu_R^c) ; \nu_L = \nu_R^c$$

$$\mathcal{L}^{D+M} = \mathcal{L}^D + \mathcal{L}_L^M + \mathcal{L}_R^M = -\frac{1}{2} \begin{pmatrix} \bar{\nu}_L^c & \bar{\nu}_R \end{pmatrix} \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix} \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix}$$

To diagonalize this mass matrix we introduce U and ρ :

$$\mathcal{L}^{D+M} = -\bar{N}_L^c M N_L ; M = \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix} N_L = \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} = U n_L \quad n_L = \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \end{pmatrix}$$

$$U^T M U = \rho^T \mathcal{O}^T M \mathcal{O} ; U \text{ is a unitary mass mixing matrix.}$$

$$\mathcal{O} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad \rho = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix} \quad \mathcal{O} = \begin{pmatrix} \rho_1 \cos \theta & \rho_2 \sin \theta \\ -\rho_1 \sin \theta & \rho_2 \cos \theta \end{pmatrix} = U$$

$$\mathcal{O}^T = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad \rho^T = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix} \quad (\mathcal{O})^T = \rho^T \mathcal{O}^T \begin{pmatrix} \rho_1 \cos \theta & -\rho_2 \sin \theta \\ \rho_1 \sin \theta & \rho_2 \cos \theta \end{pmatrix} = U^T$$

$$\mathcal{L} = \frac{1}{2} \begin{pmatrix} \bar{\nu}_L^c & \bar{\nu}_R \end{pmatrix} \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix} \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \bar{\nu}_L^c & \bar{\nu}_R \end{pmatrix} \begin{pmatrix} m_L \nu_L + m_D \nu_R^c \\ m_D \nu_L + m_R \nu_R^c \end{pmatrix}$$

$$= \frac{1}{2} (\bar{\nu}_L^c m_L \nu_L + \bar{\nu}_L^c m_D \nu_R^c + \bar{\nu}_R m_D \nu_L + \bar{\nu}_R m_R \nu_R^c)$$

$$\nu_L = \nu_R^c \quad \bar{\nu}_R = \bar{\nu}_L^c$$

$$\mathcal{L} = \frac{1}{2} \bar{\nu}_L^c m_L \nu_L + \bar{\nu}_R m_D \nu_L + \frac{1}{2} \bar{\nu}_R m_R \nu_R^c)$$

$$\text{Let : } \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} = U \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \end{pmatrix} \quad \text{Then : } \mathcal{L} = (\nu_{1L} \quad \nu_{2L}) U^T \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix} U \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \end{pmatrix}$$

To diagonalizable the mass matrix:

$$U^T \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix} U = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$$

$$U^T M U = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = \begin{pmatrix} \rho_1 \cos \theta & -\rho_1 \sin \theta \\ \rho_2 \sin \theta & \rho_2 \cos \theta \end{pmatrix} \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix} \begin{pmatrix} \rho_1 \cos \theta & \rho_2 \sin \theta \\ -\rho_1 \sin \theta & \rho_2 \cos \theta \end{pmatrix}$$

$$U^T M U = \begin{pmatrix} \rho_1 \cos \theta & -\rho_1 \sin \theta \\ \rho_2 \sin \theta & \rho_2 \cos \theta \end{pmatrix} \begin{pmatrix} m_L \rho_1 \cos \theta - m_D \rho_1 \sin \theta & m_L \rho_2 \sin \theta + m_D \rho_2 \cos \theta \\ m_D \rho_1 \cos \theta - m_R \rho_1 \sin \theta & m_D \rho_2 \sin \theta + m_R \rho_2 \cos \theta \end{pmatrix}$$

$$U^T M U = \begin{pmatrix} \rho_1 \cos \theta (m_L \rho_1 \cos \theta - m_D \rho_1 \sin \theta) - \rho_1 \sin \theta (m_D \rho_1 \cos \theta - m_R \rho_1 \sin \theta) & \rho_1 \cos \theta (m_L \rho_2 \sin \theta + m_D \rho_2 \cos \theta) - \rho_1 \sin \theta (m_D \rho_2 \sin \theta + m_R \rho_2 \cos \theta) \\ \rho_2 \sin \theta (m_L \rho_1 \cos \theta - m_D \rho_1 \sin \theta) + \rho_2 \cos \theta (m_D \rho_1 \cos \theta - m_R \rho_1 \sin \theta) & \rho_2 \sin \theta (m_L \rho_2 \sin \theta + m_D \rho_2 \cos \theta) + \rho_2 \cos \theta (m_D \rho_2 \sin \theta + m_R \rho_2 \cos \theta) \end{pmatrix}$$

$$(1) \quad \rho_1 \cos \theta (m_L \rho_1 \cos \theta - m_D \rho_1 \sin \theta) - \rho_1 \sin \theta (m_D \rho_1 \cos \theta - m_R \rho_1 \sin \theta) = m_1$$

$$(2) \quad \rho_2 \sin \theta (m_L \rho_2 \sin \theta + m_D \rho_2 \cos \theta) + \rho_2 \cos \theta (m_D \rho_2 \sin \theta + m_R \rho_2 \cos \theta) = m_2$$

$$(3) \quad \rho_1 \cos \theta (m_L \rho_2 \sin \theta + m_D \rho_2 \cos \theta) - \rho_1 \sin \theta (m_D \rho_2 \sin \theta + m_R \rho_2 \cos \theta) = 0$$

$$(4) \quad \rho_2 \sin \theta (m_L \rho_1 \cos \theta - m_D \rho_1 \sin \theta) + \rho_2 \cos \theta (m_D \rho_1 \cos \theta - m_R \rho_1 \sin \theta) = 0$$

$$(1) \quad \rho_1^2 \{m_L \cos^2 \theta - 2m_D \cos \theta \sin \theta + m_R \sin^2 \theta\} = m_1$$

$$(2) \quad \rho_2^2 \{m_L \sin^2 \theta + 2m_D \cos \theta \sin \theta + m_R \cos^2 \theta\} = m_2$$

$$(3) \quad \rho_1 \rho_2 \{m_L \sin \theta \cos \theta + m_D (\cos^2 \theta - \sin^2 \theta) - m_R \sin \theta \cos \theta\} = 0$$

$$(4) \quad \rho_1 \rho_2 \{m_L \sin \theta \cos \theta + m_D (\cos^2 \theta - \sin^2 \theta) - m_R \sin \theta \cos \theta\} = 0$$

$$(1) + (2) \Rightarrow (m_L + m_R) = \frac{m_1}{\rho_1^2} + \frac{m_2}{\rho_2^2}$$

$$(3) = (4)$$

$$(3) \quad \frac{m_L}{2} \sin 2\theta + m_D \cos 2\theta - \frac{m_R}{2} \sin 2\theta = 0$$

$$(3) \quad \frac{m_L}{2} \tan 2\theta + m_D - \frac{m_R}{2} \tan 2\theta = 0$$

$$(3) \Rightarrow \tan 2\theta = \frac{2m_D}{m_R - m_L} = x$$

$$\tan^2 2\theta = \frac{\sin^2 2\theta}{\cos^2 2\theta} = \frac{1}{\cos^2 2\theta} - 1 = x^2$$

$$\cos^2 2\theta = \frac{1}{1+x^2} = \frac{(m_R - m_L)^2}{(m_R - m_L)^2 + 4m_D^2}$$

$$(1) \rho_1^2 \{m_L \cos^2 \theta - 2m_D \cos \theta \sin \theta + m_R \sin^2 \theta\} = m_1$$

$$(2) \rho_2^2 \{m_L \sin^2 \theta + 2m_D \cos \theta \sin \theta + m_R \cos^2 \theta\} = m_2$$

$$(3) \rho_1 \rho_2 \{m_L \sin \theta \cos \theta + m_D (\cos^2 \theta - \sin^2 \theta) - m_R \sin \theta \cos \theta\} = 0$$

$$(4) \rho_1 \rho_2 \{m_L \sin \theta \cos \theta + m_D (\cos^2 \theta - \sin^2 \theta) - m_R \sin \theta \cos \theta\} = 0$$

$$(1) + (2) \Rightarrow (m_L + m_R) = \frac{m_1}{\rho_1^2} + \frac{m_2}{\rho_2^2}$$

$$(3) = (4)$$

$$(3) \frac{m_L}{2} \sin 2\theta + m_D \cos 2\theta - \frac{m_R}{2} \sin 2\theta = 0$$

$$(3) \frac{m_L}{2} \tan 2\theta + m_D - \frac{m_R}{2} \tan 2\theta = 0$$

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$$\tan^2 2\theta = \frac{\sin^2 2\theta}{\cos^2 2\theta} = \frac{1}{\cos^2 2\theta} - 1 = x^2$$

$$\cos^2 2\theta = \frac{1}{1+x^2} = \frac{(m_R - m_L)^2}{(m_R - m_L)^2 + 4m_D^2} ; \sin^2 2\theta = \frac{x}{1+x^2} = \frac{4m_D^2}{(m_R - m_L)^2 + 4m_D^2}$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$$

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2} ; \sin^2 \theta = \frac{1 - \cos 2\theta}{2}$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\begin{aligned}
(1) \quad & \rho_1^2 \{m_L \cos^2 \theta - 2m_D \cos \theta \sin \theta + m_R \sin^2 \theta\} = m_1 \\
(2) \quad & \rho_2^2 \{m_L \sin^2 \theta + 2m_D \cos \theta \sin \theta + m_R \cos^2 \theta\} = m_2 \\
(1) + (2) \Rightarrow & (m_L + m_R) = \frac{m_1}{\rho_1^2} + \frac{m_2}{\rho_2^2} \\
\cos 2\theta = & \frac{(m_R - m_L)}{\sqrt{(m_R - m_L)^2 + 4m_D^2}} \\
\sin 2\theta = & \frac{2m_D}{\sqrt{(m_R - m_L)^2 + 4m_D^2}} \\
\sin^2 \theta = & \frac{1 - \cos 2\theta}{2}; \cos^2 \theta = \frac{\cos 2\theta + 1}{2} \\
(2) \quad & m_L \sin^2 \theta + 2m_D \cos \theta \sin \theta + m_R \cos^2 \theta = \frac{m_2}{\rho_2^2} \\
(2) \quad & m_L \left(\frac{1 - \cos 2\theta}{2} \right) + m_D \sin 2\theta + m_R \left(\frac{\cos 2\theta + 1}{2} \right) = \frac{m_2}{\rho_2^2} \\
(2) \quad & (m_R - m_L) \left(\frac{\cos 2\theta}{2} \right) + m_D \sin 2\theta + \left(\frac{m_R + m_L}{2} \right) = \frac{m_2}{\rho_2^2} \\
(2) \quad & \frac{(m_R - m_L)}{2} \left(\frac{(m_R - m_L)}{\sqrt{(m_R - m_L)^2 + 4m_D^2}} \right) + m_D \frac{2m_D}{\sqrt{(m_R - m_L)^2 + 4m_D^2}} + \left(\frac{m_R + m_L}{2} \right) = \frac{m_2}{\rho_2^2} \\
(2) \quad & \left(\frac{(m_R - m_L)^2}{\sqrt{(m_R - m_L)^2 + 4m_D^2}} \right) + \frac{4m_D^2}{\sqrt{(m_R - m_L)^2 + 4m_D^2}} + (m_R + m_L) = \frac{2m_2}{\rho_2^2} \\
(2) \quad & \frac{2m_2}{\rho_2^2} = (m_R + m_L) + \sqrt{(m_R - m_L)^2 + 4m_D^2}
\end{aligned}$$

(1) $m_L \cos^2 \theta - 2m_D \cos \theta \sin \theta + m_R \sin^2 \theta = \frac{m_1}{\rho_1^2}$
 (1) $m_L \left(\frac{\cos 2\theta + 1}{2} \right) - m_D \sin 2\theta + m_R \left(\frac{1 - \cos 2\theta}{2} \right) = \frac{m_1}{\rho_1^2}$
 (1) $(m_L - m_R) \left(\frac{\cos 2\theta}{2} \right) - m_D \sin 2\theta + \left(\frac{m_R + m_L}{2} \right) = \frac{m_1}{\rho_1^2}$
 (1) $\frac{(m_L - m_R)}{2} \left(\frac{(m_R - m_L)}{\sqrt{(m_R - m_L)^2 + 4m_D^2}} \right) - m_D \frac{2m_D}{\sqrt{(m_R - m_L)^2 + 4m_D^2}} + \left(\frac{m_R + m_L}{2} \right) = \frac{m_1}{\rho_1^2}$
 (1) $\left(\frac{-(m_R - m_L)^2}{\sqrt{(m_R - m_L)^2 + 4m_D^2}} \right) - \frac{4m_D^2}{\sqrt{(m_R - m_L)^2 + 4m_D^2}} + (m_R + m_L) = \frac{2m_1}{\rho_1^2}$
 (1) $\frac{2m_1}{\rho_1^2} = (m_R + m_L) - \left(\frac{(m_R - m_L)^2 + 4m_D^2}{\sqrt{(m_R - m_L)^2 + 4m_D^2}} \right)$
 (1) $\frac{2m_1}{\rho_1^2} = (m_R + m_L) - \sqrt{(m_R - m_L)^2 + 4m_D^2}$

$$m_{1,2} = \frac{\rho_{1,2}^2}{2} \left\{ (m_R + m_L) \pm \sqrt{(m_R - m_L)^2 + 4m_D^2} \right\}$$

$$m_{1,2} = \frac{\rho_{1,2}^2}{2} \left\{ (m_R + m_L) \pm (m_R - m_L) \sqrt{1 + \frac{4m_D^2}{(m_R - m_L)^2}} \right\}$$

if $m_R + m_L \gg 2m_D$ then :

$$m_{1,2} = \frac{\rho_{1,2}^2}{2} \left\{ (m_R + m_L) \pm (m_R - m_L) \left(1 + \frac{4m_D^2}{2(m_R - m_L)^2} \right) \right\}$$

$$m_1 = \rho_1^2 \left\{ m_R + \left(\frac{m_D^2}{(m_R - m_L)} \right) \right\}$$

$$m_2 = \rho_2^2 \left\{ m_L - \left(\frac{m_D^2}{(m_R - m_L)} \right) \right\}$$

The left handed majorana neutrino must not exist and $\rho_{1,2} = 1$

$$m_1 \approx m_R$$

$$m_2 \approx \left(\frac{m_D^2}{m_R} \right)$$